A.A. Dezin
Partial Differential Equations

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An Introduction to a General Theory of Linear Boundary Value Problems

Translated from the Russian
by Ralph P. Boas

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Preface

Let me begin by explaining the meaning of the title of this book. In essence, the book studies boundary value problems for linear partial differential equations in a finite domain in n-dimensional Euclidean space. The problem that is investigated is the question of the dependence of the nature of the solvability of a given equation on the way in which the boundary conditions are chosen, i.e. on the supplementary requirements which the solution is to satisfy on specified parts of the boundary.

The branch of mathematical analysis dealing with the study of boundary value problems for partial differential equations is often called mathematical physics.

Classical courses in this subject usually consider quite restricted classes of equations, for which the problems have an immediate physical context, or generalizations of such problems.

With the expanding domain of application of mathematical methods at the present time, there often arise problems connected with the study of partial differential equations that do not belong to any of the classical types. The elucidation of the correct formulation of these problems and the study of the specific properties of the solutions of similar equations are closely related to the study of questions of a general nature.

Among these are the following:

1. What accounts for the special position of the classical equations of mathematical physics (and their generalizations) among all possible equations?

2. Can one find a reasonable (in some sense of this term) boundary value problem for a randomly chosen equation, and if so, how?

3. What is the nature of the pathological phenomena that arise in the case of incorrectly posed boundary value problems?

These questions, and similar ones, need, of course, to be clarified, and are far from having complete answers. Nevertheless, it is clear that they should not be assumed to be merely speculative. The ability to orient one's self in unconventional situations is often valuable for a mathematician or physicist who is concerned with the solution of specific problems. For this reason, the author has tried to make the book accessible to the widest possible circle of readers.

Boundary value problems for partial differential equations constitute a rich and complicated subject, and can be considered from very diverse points of view. The basic approach in this book is through the theory of linear operators in Hilbert space. In certain constructions we also use spaces with other structures, but the Hilbert space of functions of integrable square is fundamental. In this connection, it is frequently most convenient to formulate the solvability properties of a boundary value problem in terms of the properties of the spectrum of an operator associated with the problem.

The first (introductory) chapter "Elements of spectral theory" is a brief exposition of the necessary facts from the corresponding parts of functional analysis.

In the second chapter we discuss general methods of associating a boundary value problem with a linear operator on Hilbert space.

The generality of the questions enumerated above makes it necessary to impose a number of quite stringent restrictions on the operators that we shall study. The elucidation of correct formulations of problems and the study of particular properties of their solutions for "nonclassical" equations is conveniently begun by the consideration of idealized models, for example by considering equations with constant coefficients, with part of the boundary conditions replaced by the condition of periodicity. This allows the application of some version of the method of separation of variables. In essence, the main part of the book (Chapters IV–VI) is based on the use of methods of this kind. By means of these we are led to the consideration of special classes of operator equations for which it is possible to obtain meaningful and rather complete results.

The reader can obtain additional details about the content of the book by looking through it. Numerous general remarks are contained in the introductory subsections, numbered "0".

In conclusion, I offer the following additional remarks. If the books in which the methods of functional analysis are applied to the study of boundary value problems are conditionally divided into two groups:

1) treatises on functional analysis in which differential operators are studied as concrete examples;

2) treatises on the theory of partial differential equations in which functional analysis is one of the methods employed;

then, putting this monograph into the second group, I would emphasize that my intention is that the basic theme should be an exposition of the mechanism of applying the general concepts of functional analysis to the study of definite classes of specific classical entities.

In conclusion, I take this opportunity to thank Professor Sh.A. Alimov for reading the manuscript and making many valuable comments.

A.A. Dezin

Preface to the English Edition

The main theme of this book is the study of how the solvability of a given linear partial differential equation depends on the choice of the boundary conditions; the principal methods are those of functional analysis. I feel that this theme deserves more attention than it usually receives. Rather than proving many general theorems, I have presented numerous special cases, for which more or less complete results are attainable, in order to illustrate various kinds of results. I hope that these examples will help the reader acquire enough intuition so that they can analyze the particular problems that arise in their own work. For a fuller discussion of the objectives of the book, the reader is referred to the preface to the Russian edition (above).

Shortly after the publication of the first edition, an approach was discovered to many of the problems that are discussed in the main part of the book; it is known as the model-operator method. It has become clear that with this approach one can analyze a large class of diverse problems, both from a unified point of view and in simplified formulations. A number of results in this direction are outlined in an appendix that contains brief summaries, kindly provided by Professor Boas, of some recent papers.

In conclusion, I want to express my gratitude to Professor Boas and to Springer-Verlag for producing this English edition, which should make the book accessible to a wider circle of readers.

Moscow, December 1986 A.A. Dezin

To the Reader

The book is divided into chapters; the chapters, into sections; the sections, into subsections. Formulas, theorems, and statements are numbered within each section. For a reference within a section, the number is given; for a reference to a different section of the same chapter, also the section (or section and subsection). Otherwise the chapter is also given.

Numbers in square brackets are references to the corresponding books or papers in the bibliography. A reference does not imply that the book or paper cited is the only (or principal) source of the information in question.

The "Halmos symbol" □ marking the end of a proof (possibly only an outline), or to emphasize its absence, is not used altogether systematically. In some cases where no confusion will result, it is omitted.

Definitions are not always set off in separate paragraphs. Frequently they are run into the text. Definitions of concepts are printed in italics.

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Chapter I
Elements of Spectral Theory

§0. Introductory Remarks

This chapter is introductory in nature. It contains the basic facts from the theory of linear operators that are fundamental for what follows. The contents of the chapter are more than amply covered by standard textbooks, for example those listed in the bibliography. Exact references are, in some cases, given in the text.

The reasons for including such a chapter in the book are evident: it is always convenient to have a brief summary of the information that is assumed to be known. Such a summary should eliminate the possibility of terminological discrepancies and serve the inexperienced reader as a compass in navigating through the ocean of propositions formed by the contents of the often terrifying volume of courses in functional analysis.

It is rather more difficult to justify the presence (or absence) of proofs. This is all the more true since the proofs that are given are sometimes quite detailed, whereas others are in the nature of hints. It is clear that the presence of proofs always gives a plan for more complete ones. Moreover, sometimes a proof lets us make a remark that seems to the author to be important; sometimes, it indicates a useful technical device; and sometimes the aim of a proof is simply to lighten the task of a reader who really wants everything to be proved.

Among the essential remarks mentioned above there belongs a mention of the details of our point of view, which is dictated by the fundamental subject of our study: the boundary value problem. Not all the facts enumerated in this chapter are used to the same extent. Some are presented only to complete the picture which will serve as general background to later constructions.

There are no examples in this chapter. The whole remainder of the discussion will serve as a set of examples for it.

§1. Basic Definitions

1.0. Introductory Remarks. The natural framework for the "abstract" spectral theory of operators, i.e. a theory that does not specify the way in which the operators are defined, is a complex Banach space. Although we shall be mainly concerned with a specific function space, namely Hilbert space, it is natural to present some facts in a more general setting. Moreover, it is just in this setting that they are presented in the standard treatises.

It should be noted that when one is concerned with "spectral theory" rather than "spectral theory of operators," contemporary references take as fundamental object an element of a certain Banach algebra. Hence the point of view of the present chapter may appear not to be abstract at all, but rather "concrete."

1.1. Fundamental Structure. If we start from the initial concepts of "naive set theory" - sets and relations, and follow the chain of axioms that enter into the definition of a Banach space, we obtain the following picture.

An *Abelian group* is a nonempty set G of elements $a, b, c, \ldots$, with a binary operation "+" that associates with every pair a, b of elements of G a unique element $c \in G$ $(a+b=c)$. The operation "+" is subject to the following additional requirements: it is associative $((a+b)+c=a+(b+c))$, commutative $(a+b=b+a)$; there is a neutral element 0 $(a+0=a)$; and for every $a \in G$ there is an inverse element $-a$ such that $a+(-a)=0$.

A *complex linear space* $\mathscr{K}$ is an Abelian group in which there is defined a multiplication of elements $a, b, c, \ldots$ by complex numbers $\alpha, \beta, \gamma, \ldots$, such that the following conditions are satisfied:

$$\alpha(a+b)=\alpha a+\alpha b, \quad (\alpha+\beta) a=\alpha a+\beta a,$$
$$(\alpha \beta) a=\alpha(\beta a), \quad 1a=a.$$

If we replace the complex numbers by the real numbers, we obtain the definition of a *real* linear space.

We emphasize that in restricting the class of *numbers* $\alpha, \beta, \gamma, \ldots$ in these definitions we are considering our objects from the point of view of *analysis*. An algebraist would have allowed the elements $\alpha, \beta, \gamma, \ldots$ in the definition to belong to an arbitrary field $\mathscr{F}$.

A *norm* is a nonnegative real function $\|a\|$, defined on elements $a \in \mathscr{K}$ and satisfying the following conditions:

1) $\|a\|=0$ implies $a=0$,
2) $\|\alpha a\|=|\alpha|\,\|a\|$,
3) $\|a+b\| \leq \|a\|+\|b\|$.

A space $\mathscr{K}$ that has a norm is called a *normed linear space* (the qualifier "complex" or "real" will usually be omitted).

A sequence $\{x_n\}$ of elements of $\mathscr{K}$ is a *Cauchy sequence* if for every $\varepsilon>0$ there is an integer $N(\varepsilon)$ such that the condition $m, n>N$ implies $\|x_n-x_m\| <\varepsilon$. A space $\mathscr{K}$ is complete if for every Cauchy sequence there is an element $x \in \mathscr{K}$ to which this sequence converges (in the ordinary sense).

A complete normed linear space is called a *Banach space* (B-space).

In a linear space the concept of linear dependence is defined in the usual way, and consequently so is the concept of *dimension:* the largest number of linearly independent elements. Although the spaces in which we are interested will usually be infinite-dimensional, we do not include infinite dimensionality in the definition of a B-space. Thus the set of complex numbers with the modulus as norm is an example of a one-dimensional B-space.

An incomplete normed linear space is called a *pre-Banach* space. Every pre-Banach space can be extended to a Banach space by the abstract process of adjoining the limits of the convergent sequences ([12], Chap. II, § 3.4).

Every B-space is simultaneously both a metric space and a topological space, but this aspect is without interest for our purposes.

A complex linear space $\mathscr{H}$ is a *pre-Hilbert* space if to every ordered pair of elements a, b there is assigned a complex number (a, b), their scalar product, satisfying the following requirements:

1) $(a, a) \geq 0$ and $(a, a) = 0$ implies $a = 0$;
2) $(a, b) = \overline{(b, a)}$ (the bar denotes the complex conjugate);
3) $(a + b, c) = (a, c) + (b, c)$;
4) $(\alpha a, b) = \alpha (a, b)$.

If in a pre-Hilbert space we set $\|a\|^2 = (a, a)$, it follows immediately from the classical Bunyakovsky-Schwarz inequality

$$|(a, b)| \leq \|a\| \, \|b\|$$

that the function $\|a\|$ is a norm, and therefore a pre-Hilbert space is automatically a pre-Banach space.

The scalar product is continuous: $\lim_k (a_k, b) = (\lim_k a_k, b)$.

A complete normed pre-Hilbert space is called a *Hilbert space*. Every Hilbert space is a B-space. In order for it to be possible to introduce a scalar product that generates a norm in a Banach space, certain special requirements have to be satisfied ([21], Chap. I, § 5; in that book the term "pre-Hilbert space" has a quite different meaning).

1.2. Special Subsets. A subset $\mathscr{B}'$ of a Banach space $\mathscr{B}$ which is in turn a B-space with the norm induced by then norm of $\mathscr{B}$ is called a *subspace* of $\mathscr{B}$.

We are often led to encounter a subset $\mathscr{B}' \subset \mathscr{B}$ which is a linear subspace but does not satisfy the condition of completeness in the norm of $\mathscr{B}$. We call such a subset a *linear manifold*.

The simplest way to form a linear manifold in $\mathscr{B}$ is to consider the *linear span* of a given subset $\mathscr{M} \subset \mathscr{B}$, i.e. the set of all finite linear combinations of elements of $\mathscr{M}$. If $\mathscr{B}'$ also contains all limit elements, i.e. limits (in the norm of $\mathscr{B}$) of sequences of elements of $\mathscr{M}$, then the corresponding *closed* linear span is a subspace of $\mathscr{B}$. This distinction naturally occurs only when $\mathscr{M}$ has an infinite number of linearly independent elements.

A subset $\mathscr{Q} \subset \mathscr{B}$ is *dense* in $\mathscr{B}$ if its closure is $\mathscr{B}$. A set $\mathscr{M}$ of elements is *complete* in $\mathscr{B}$ if the linear span of $\mathscr{M}$ is dense. A complete set $\{e_k\}$ of elements of $\mathscr{B}$ (finite or countable) forms a *basis* if, in the representation of each element, $x = \sum_k x_k e_k$, the numbers x_k are uniquely determined. Although there are many important examples of B-spaces without countable bases, we shall not encounter them here.

Turning now to a Hilbert space of more particular interest to us, we first notice the following fundamental proposition.

Lemma (on orthogonal expansion). *Let $\mathscr{M}'$ be a linear manifold in the Hilbert space $\mathscr{H}$, and let $\mathscr{N}$ be the set of elements $\varphi \in \mathscr{H}$ such that $(\varphi, y) = 0$ for every $y \in \mathscr{M}'$. Then $\mathscr{N}$ is a subspace of $\mathscr{H}$, and each $x \in \mathscr{H}$ has a unique representation of the form*

$$x = x_{\mathscr{M}} \oplus x_{\mathscr{N}} \tag{1}$$

where $x_{\mathscr{M}} \in \mathscr{M}$ (the closure of $\mathscr{M}'$) and $x_{\mathscr{N}} \in \mathscr{N}$.

Remark. The subspace $\mathscr{N}$ is called the *orthogonal complement* of $\mathscr{M}$ (or $\mathscr{M}'$), and (1) is the *orthogonal expansion* of x. The symbol $\oplus$ indicates this, and is also used in the notation $\mathscr{H} = \mathscr{M} \oplus \mathscr{N}$.

Proof of the lemma. We evidently need to consider only the case when $\mathscr{H}$ is infinite. In that case, that $\mathscr{N}$ is a subspace follows immediately from the properties of the scalar product.

If $x \in \mathscr{M}$, the proposition is trivial. Let $x \notin \mathscr{M}$ and let $\inf_{y \in \mathscr{M}} \|x - y_n\| = d$. Then there is a sequence $\{y_n\}$ such that $\|x - y\| = d_n \to d$ as $n \to \infty$. Let us show that the sequence $\{y_n\}$ converges, i.e. the infimum is attained for some element $y \in \mathscr{M}$. Using the definition of the norm in $\mathscr{H}$ as the square of the scalar product, we obtain

$$d_m^2 + d_n^2 = \|x - y_m\|^2 + \|x - y_n\|^2 = \tfrac{1}{2}(\|2x - y_m - y_n\|^2 + \|y_m - y_n\|^2). \tag{2}$$

Since

$$2\left\|x - \frac{y_m + y_n}{2}\right\|^2 \geq 2d^2,$$

we have the inequality

$$d_n^2 + d_m^2 - 2d^2 \geq \tfrac{1}{2}\|y_m - y_n\|^2,$$

which shows that the sequence $\{y_n\}$ converges to some $y \in \mathscr{M}$.

Let us show that $x - y \in \mathscr{N}$. In fact, the function F of the real parameter t,

$$F(t) = \|x - y + t z\|^2,$$

must have, for an arbitrarily given element $z \in \mathcal{M}$, a minimum for $t=0$, i.e. $F'(0)=0$. Hence (considering the pair z, iz of vectors) we conclude that $(x-y, z)=0$ for every $z \in \mathcal{M}$. Setting $x_{\mathcal{M}}=y$, $x_{\mathcal{N}}=x-y$, we obtain (1). The uniqueness of the representation is evident. □

The preceding proof can be instructively "geometrized" (in an especially intuitive way in a real Hilbert space). As is easily seen, the discussion remains valid if the space $\mathcal{M}$ is replaced by any closed convex set (one for which $y_1, y_2 \in \mathcal{M}$ implies $(y_1+y_2)/2 \in \mathcal{M}$). The element $x-y$ is said to be the perpendicular from x to $\mathcal{M}$; the chain of inequalities (2) uses the classical relationship between a diagonal and a side of a parallelogram. The existence of this property is a characteristic property of the Hilbert norm mentioned above. The nontrivial verification of the existence of the element y for which the infimum is attained is a consequence of the infinite dimensionality.

From the lemma we at once obtain the following corollary.

Corollary. *A set* $\mathcal{M} \subset \mathcal{H}$ *is complete if and only if the equation* $(y, x)=0$ *for every* $y \in \mathcal{M}$ *implies* $x=0$. □

We can apply the classical process of orthogonalization to any countable basis $\{\varphi_k\}$ in Hilbert space, and thereby obtain an *orthonormal basis* $\{e_k\}$ that satisfies the conditions $(e_k, e_j)=\delta_{kj}$ (the Kronecker symbol). Then the coefficients of the expansion $x=\sum_k x_k e_k$ of an element $x \in \mathcal{H}$ are determined by the equations $x_k=(x, e_k)$. An orthonormal system $\{e_k\}$ is a basis if and only if

$$\|x\|^2=\sum_k |(x, e_k)|^2 \tag{3}$$

for every $x \in \mathcal{M}$.

Remark. By using the availability of a countable orthonormal basis in $\mathcal{H}$ we can obtain a shorter (but less instructive) proof of the lemma on orthogonal expansions.

If $\{\varphi_k\}$ is a basis in $\mathcal{M}$, there exists a uniquely determined system of elements $\{\psi_k\}$ such that $(\varphi_k, \psi_j)=\delta_{kj}$. The system $\{\psi_k\}$ is also a basis, and is said to be *biorthogonal* to $\{\varphi_k\}$. For a pair of conjugate biorthogonal bases the coefficients of the expansions $x=\sum_k x_k \varphi_k$ and $y=\sum_k y_k \psi_k$ are determined by the formulas $x_k=(x, \psi_k)$, $y_k=(y, \varphi_k)$.

A basis $\{\varphi_k\}$ in $\mathcal{H}$ is called a *Riesz basis* if there are constants c_1, c_2 such that, for every $x \in \mathcal{H}$ that is represented in the form $x=\sum_k x_k \varphi_k$, we have the inequalities

$$c_1 \sum_k |x_k|^2 \le \|x\|^2 \le c_2 \sum_k |x_k|^2. \tag{4}$$

Inequalities (4) serve as a replacement for (3) in cases when the latter is not available.

1.3. Operators. A function $\mathbf{T}$ defined on a set $\mathfrak{D}(\mathbf{T}) \subset \mathscr{B}_1$ and making each element $x \in \mathfrak{D}(\mathbf{T})$ correspond to a unique element $y = \mathbf{T}x$, $y \in \mathfrak{R}(\mathbf{T}) \subset \mathscr{B}_2$, where $\mathscr{B}_1$ and $\mathscr{B}_2$ are B-spaces, is usually called an *operator*. The sets $\mathfrak{D}(\mathbf{T})$ and $\mathfrak{R}(\mathbf{T})$ are called the *domain* and the *range* of $\mathbf{T}$.

We shall consider only *linear* operators $\mathbf{T}$, i.e. operators that satisfy

$$\mathbf{T}(\alpha x + \beta y) = \alpha \mathbf{T} x + \beta \mathbf{T} y \tag{5}$$

for all numbers α, β and elements $x, y \in \mathfrak{D}(\mathbf{T})$.

Besides $\mathfrak{D}(\mathbf{T})$ and $\mathfrak{R}(\mathbf{T})$, the most important set associated with $\mathbf{T}$ is $N(\mathbf{T}) = \operatorname{Ker} \mathbf{T}$, the *kernel* of $\mathbf{T}$, i.e. the set of $x \in \mathfrak{D}(\mathbf{T})$ such that $\mathbf{T}x = 0$. It follows at once from (5) that the sets $\mathfrak{D}$, $\mathfrak{R}$ and N are linear manifolds.

An operator $\mathbf{T}^{-1}$: $\mathscr{B}_2 \to \mathscr{B}_1$ (read, "acting from $\mathscr{B}_2$ to $\mathscr{B}_1$") is called an *inverse* of $\mathbf{T}$ if $\mathbf{T}^{-1}\mathbf{T} = E$ (the identity mapping) on $\mathfrak{D}(\mathbf{T})$. A necessary and sufficient condition for the existence of $\mathbf{T}^{-1}$ is evidently that $N(\mathbf{T}) = 0$ (an operator $\mathbf{T}^{-1}$ defined in this way is sometimes called a *left* inverse).

The *norm* of $\mathbf{T}$ is $\|\mathbf{T}\| = \sup\limits_{x \in \mathfrak{D}(\mathbf{T})} (\|\mathbf{T}x\|_2 / \|x\|_1)$ (the norms of x and $\mathbf{T}x$ are the norms in $\mathscr{B}_1$ and $\mathscr{B}_2$, respectively). An operator $\mathbf{T}$ is *bounded* if its norm is finite ($\|\mathbf{T}\| < \infty$). The following fact is an important consequence of the linearity of $\mathbf{T}$.

Lemma. *The operator* $\mathbf{T}$ *is bounded if and only if it is continuous, i.e. if a sequence* $\{x_n\}$ *in* $\mathscr{B}_1$ *converges* ($x_n \to x$) *then the sequence* $\mathbf{T}x_n$ *in* $\mathscr{B}_2$ *also converges* ($\mathbf{T}x_n \to \mathbf{T}x$).

An unbounded linear operator $\mathbf{T}$, considered within the framework of normed linear spaces, is, in a certain sense, a pathological object: its definition is only "weakly compatible" with the fundamental structure, the norm. The difficulties connected with the study of operators generated by differentiation are closely related to the fact that in the most "convenient" function spaces this study inevitably leads to the consideration of unbounded operators. The basic method of overcoming these difficulties involves the use of the boundedness of the inverse of a given operator and the introduction of a notion of closure that is weaker than boundedness (continuity).

An operator $\mathbf{T}$: $\mathscr{B}_1 \to \mathscr{B}_2$ is *closed* if $x_n \to x$ and $\mathbf{T}x_n \to f$ imply that $x \in \mathfrak{D}(\mathbf{T})$ and $\mathbf{T}x = f$.

A bounded operator (extended, in a natural way, by continuity; see below) is always closed. The converse is in general false. Nevertheless we have the following proposition.

Theorem (Banach). *A closed operator whose domain is the whole space* $\mathscr{B}_1$ *is bounded.*

This theorem, which is one form of the closed graph theorem, has many aspects. The reader may reduce its statement to the form given above, in which we shall use it.

In reformulating boundary value problems in the language of operator theory, we inevitably have to use one or another generalization of the solution of the equation under consideration, a generalization obtained by extending the domain of the operator of differentiation. Let us give an abstract version of such a procedure.

An operator $\tilde{\mathbf{T}}$ is an *extension* of the operator $\mathbf{T}$, $\mathbf{T}\colon \mathscr{B}_1 \to \mathscr{B}_2$, if $\mathfrak{D}(\mathbf{T}) \subset \mathfrak{D}(\tilde{\mathbf{T}})$ and both operators coincide on $\mathfrak{D}(\mathbf{T})$.

Standard examples of the use of extensions are the extension by continuity to the whole space of a bounded operator with dense domain, and the closure of a given operator $\mathbf{T}$, i.e. the construction of the minimal closed extension $\tilde{\mathbf{T}} \supset \mathbf{T}$ (if there is such an extension). As we observed above, we shall make extensive use of the fact that operators generated by differentiation have closed extensions.

The last definition in this subsection applies only to Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$. Let $\mathbf{T}\colon \mathscr{H}_1 \to \mathscr{H}_2$ and let the element $y \in \mathscr{H}_2$ have the property that there is an element $h \in \mathscr{H}_1$ such that the equation

$$(\mathbf{T}x, y)_2 = (x, h)_1$$

is satisfied for every $x \in \mathfrak{D}(\mathbf{T})$ (scalar products in $\mathscr{H}_2$ and $\mathscr{H}_1$, respectively). We then define the operator $\mathbf{T}^*$ (the *adjoint of* $\mathbf{T}$) by $\mathbf{T}^*\colon \mathscr{H}_2 \to \mathscr{H}_1$, by setting $y \in \mathfrak{D}(\mathbf{T}^*)$, $\mathbf{T}^* y = h$ (under the conditions specified above). This definition is consistent if and only if $\mathfrak{D}(\mathbf{T})$ is dense in $\mathscr{H}_1$. Indeed, this condition guarantees that h is uniquely determined; $\mathbf{T}^*$ is evidently linear, and $\mathfrak{D}(\mathbf{T}^*)$ contains at least the zero element.

The following consequence of this definition is especially useful.

Proposition 1. *An operator* $\mathbf{T}^*$ *is closed if it is the adjoint of some operator* $\mathbf{T}$.

In fact, if $y_n \to y$ and $\mathbf{T}^* y_n = h_n \to h$, we may take limits in the equation $(\mathbf{T}x, y_n) = (x, h_n)$ because of the continuity of the scalar product. □

Another useful proposition is conveniently stated in algebraic language. We draw the picture (known as a diagram) where the horizontal lines indicate passage to the adjoint operator; and the vertical lines, to the inverse.

$$\begin{array}{ccc} \mathbf{T} & \xrightarrow{\;*\;} & \mathbf{T}^* \\ \big\downarrow \text{Inv} & & \big\downarrow \text{Inv} \\ \mathbf{T}^{-1} & \xrightarrow{\;*\;} & \mathbf{T}^{-1*} \end{array}$$

Proposition 2. *If, for a given operator* $\mathbf{T}\colon \mathscr{H}_1 \to \mathscr{H}_2$, *we define the operations indicated on the diagram by the solid lines, then the dotted line makes the diagram commute, i.e. the operator* $\mathbf{T}^{*-1}$ *exists and* $\mathbf{T}^{*-1} = \mathbf{T}^{-1*}$.

Proof. The operator $\mathbf{T}^{*-1}$ exists. In fact, if the equation $(\mathbf{T}u, v)=(\mathbf{T}u, w)$ is satisfied for every $u\in\mathfrak{D}(\mathbf{T})$ then since $\mathfrak{R}(\mathbf{T})$ is dense in $\mathscr{H}_2$ (otherwise $\mathbf{T}^{*-1}$ would not exist) we have $v=w$, i.e. $N(\mathbf{T}^*)=0$.

Let us establish the inclusion $\mathbf{T}^{*-1}\subset\mathbf{T}^{-1*}$. Let $u\in\mathfrak{D}(\mathbf{T}^{*-1})$, and let $u=\mathbf{T}^*v$. Then for every $w\in\mathfrak{D}(\mathbf{T})$ we have the equation

$$(\mathbf{T}w, v)=(w, u). \tag{6}$$

If now $\mathbf{T}w=h$, $w=\mathbf{T}^{-1}h$, then (6) yields

$$(\mathbf{T}^{-1}h, u)=(h, v),$$

i.e. $u\in\mathfrak{D}(\mathbf{T}^{-1*})$ and $\mathbf{T}^{-1*}u=v=\mathbf{T}^{*-1}u$.

The converse inclusion is verified by similar reasoning. □

1.4. Functionals. A linear operator $\mathscr{L}$: $\mathscr{B}\to\mathbb{C}$, where $\mathbb{C}$ is the Banach space of complex numbers ($\mathbb{C}$ may also be considered as a Hilbert space with the scalar product $(\alpha, \beta)=\alpha\bar{\beta}$) is called a *functional* (or *complex functional* to distinguish it from the *real* functionals $\mathscr{L}$: $\mathscr{B}\to\mathbb{R}$).

Since functionals are operators of a special kind, everything that we have said previously about operators carries over directly to functionals.

The set of all bounded functionals on a B-space $\mathscr{B}$ forms the *dual* space $\mathscr{B}^*$ of $\mathscr{B}$; this space plays a fundamental role in many situations. The special place occupied by the Hilbert space $\mathscr{H}$ among the Banach spaces is determined to a significant extent by the fact that $\mathscr{H}^*$ can be identified, in a natural way, with $\mathscr{H}$, i.e. in this sense we have selfadjointness. Let us prove the corresponding proposition.

Lemma (Riesz). *Let $\mathscr{L}$ be a bounded functional defined on a linear manifold $\mathscr{M}'\subset\mathscr{H}$. Then there is a unique element $h\in\mathscr{M}$ (the closure of $\mathscr{M}'$) such that*

$$\mathscr{L}(x)=(x, h) \tag{7}$$

for all $x\in\mathscr{M}'$. Moreover, $\|\mathscr{L}\|=\|h\|$.

Proof. We may consider $\mathscr{M}$ as a Hilbert space $\mathscr{H}_1\subset\mathscr{H}$ (with scalar product given by the product in $\mathscr{H}$) and take $\mathscr{L}$ to be defined by continuity on the whole space $\mathscr{H}_1$.

If $\mathscr{L}(x)=0$ for every $x\in\mathscr{H}_1$, we can set $h=0$. If $\mathscr{L}\neq0$, then $N(\mathscr{L})$ (the kernel of $\mathscr{L}$) is a closed subspace different from $\mathscr{H}_1$. Consider the decomposition $\mathscr{H}_1=N(\mathscr{L})\oplus\mathscr{Q}$. The subspace $\mathscr{Q}$ is one-dimensional. In fact, for every pair of elements $x_1, x_2\in\mathscr{Q}$, with $\mathscr{L}(x_1)=\alpha_1\neq0$ and $\mathscr{L}(x_2)=\alpha_2\neq0$, we have

$$\mathscr{L}\left(\frac{x_1}{\alpha_1}-\frac{x_2}{\alpha_2}\right)=0.$$

Let $q \in \mathcal{Q}$ be a basis element, $\|q\|=1$, $\mathcal{L}(q)=\beta$. Then $\mathcal{L}(x)=(x, \bar{\beta} q)$ for every $x \in \mathcal{H}_1$. In fact, let us represent x in the form $x = x_N \oplus x_{\mathcal{Q}}$, and take $x_{\mathcal{Q}} = k q$. Then

$$\mathcal{L}(x)=\mathcal{L}(x_{\mathcal{Q}})=\mathcal{L}(k q)=k \beta,$$
$$(x, \bar{\beta} q)=\beta(x, q)=k \beta(q, q)=k \beta.$$

It is evident that the element $h=\bar{\beta} q$ of $\mathcal{H}_1$ is unique. Moreover,

$$\|\mathcal{L}\|=\sup_{x \in \mathcal{H}_1} \frac{|\mathcal{L}(x)|}{\|x\|} \leq \frac{|\mathcal{L}(x)|}{\|x_{\mathcal{Q}}\|}=\frac{|k \beta(q, q)|}{|k|}=|\beta|=\|h\|,$$

and for $x \in \mathcal{Q}$ the inequality becomes an equality. □

The preceding proof is an obvious modification of the argument used for a finite-dimensional space $\mathcal{H}$. The equation $\mathcal{L}(x)=0$ defines a hyperplane. It is, in general, infinite-dimensional, but its orthogonal complement is always one-dimensional. This fact was applied in the proof.

As we shall see, Riesz's lemma is a very useful tool in proving theorems about the existence of solutions of operator equations.

Corollary. *A bounded functional defined on a linear manifold $\mathcal{M}' \subset \mathcal{H}$ can be extended, with preservation of its norm, to the whole space $\mathcal{H}$.*

In fact, formula (6) evidently provides the required extension.

For a general B-space, the preceding corollary contains the content of the Hahn-Banach theorem. Its proof is considerably more complicated since in that case we do not have an explicit general form for linear functionals.

The Hahn-Banach theorem, together with Banach's theorem (above) and the uniform boundedness principle (which we shall not have occasion to use), otherwise known as the Banach-Steinhaus theorem, are the big theorems of the classical theory of B-spaces.

§2. The Spectrum of an Operator

2.0. Preliminary Remarks. One of the most useful features of the entities considered in linear functional analysis is the existence for many of them of analogies of a much simpler nature. The existence of these analogies enriches our intuition and has great heuristic value. These often originate in parallels with the theory of finite- and infinite-dimensional spaces (here it is appropriate to mention the book [8]), i.e. with analogies with vectors (points in Euclidean spaces) or functions (points in the Hilbert space of functions), and in parallels between the algebra of complex numbers and the algebra generated by the family of commutative operators on a given B-space.